Aerospace Research Center

BIRKHOFF NORMALIZATION PROCESS PROGRAM
FOR
TIME-DEPENDENT HAMILTONIAN SYSTEMS

by

W. Fine and S. Kass
Mathematics Department

AFOSR Scientific Report No. AFOSR-68-0875

This research was supported in part by the AF Office of Scientific Research of the Office of Aerospace Research, under contract AF49 (638) 1466; in part by the NASA Marshall Space Flight Center under contract NAS 8 - 20062; and in part by the NASA Goddard Space Flight Center under contract NAS 5 - 9350.

May 1968

AEROSPACE RESEARCH CENTER
GENERAL PRECISION SYSTEMS INC.
KEARFOTT GROUP
LITTLE FALLS, NEW JERSEY

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Abstract

This is a complete documentation of a computer program for analyzing finite time stability properties of equilibrium points of time-dependent (nonconservative) Hamiltonian systems. Using this package, approximate analytic solutions which contain nonlinear effects can be constructed near these equilibrium points. Qualitative rate-of-growth estimates can also be obtained. This program description is a sequel to "Birkhoff Normalization Process Program for Time-Independent Hamiltonian Systems", AFOSR Scientific Report No. AFOSR 67-0123 (October, 1966) which contained the program description for analyzing time-independent Hamiltonian systems.

Eleven programs constitute the complete package. The first six programs perform the second order normalization and the next three apply the Birkhoff technique for higher order normalizations. These nine programs compute the generating functions and coordinate transformations for a time-dependent Hamiltonian in the neighborhood of an equilibrium point. The remaining two programs evaluate actual trajectories for the particular application under consideration. The first of these evaluates the trajectory for the normalized Hamiltonian and the second performs a point by point integration of the equations of motion of the given dynamical system.

The computer program developed here has been used to generate an algebraic solution for the motion of a particle near an equilibrium point of the planar elliptical restricted three-body problem. Sufficient time-dependent and nonlinear terms have been retained to produce trajectories which match solutions obtained by numerical integration of the equations of motion for 30 days or longer. Computation time for the normalized trajectory is insignificant compared to the time required for numerical integration. Sources of truncation and round-off errors in the normalization process are identified.

TABLE OF CONTENTS

I. INTRODUCTION

The study of stability properties of solutions of nonlinear dynamical systems presents serious mathematical difficulties. For this reason, early investigations of the stability of the triangular equilibrium points or libration points of the circular restricted three-body problem of celestial mechanics were limited to the linear case, or the first approximation [1]. This is true also in more recent studies of the elliptical restricted three-body problem* [2,3,4]. Unfortunately, the behavior of these and many other interesting nonlinear systems cannot be predicted from the linear analysis, but depends on the nature of the nonlinear terms. Theoretically, stability can be decided by the construction of a Liapunov function [5], but there is no straightforward way to construct such functions for arbitrary dynamical systems.

Considerable progress in the qualitative theory of nonlinear dynamical systems has been made following Kolmogorov in 1954, notably by V.I. Arnold and J. Moser. In 1962, Leontovich [6] applied a theorem of Arnold to establish the stability of the triangular libration points of the planar circular restricted three-body problem for almost all admissible mass ratios. Very recently, Deprit and Deprit-Bartholomé [7] have applied results of Moser to refine Leontovich's results. Corresponding work in the elliptical problem has not appeared in the literature.

G.D. Birkhoff [8] established in 1927 a quantitative method for determining stability properties of equilibrium points and periodic solutions of Hamiltonian systems. He showed that by an infinite series of steps, the differential equations can be transformed into an integrable system of equations, whose solution may be transformed into the original variables to determine the behavior of solutions near the given equilibrium point or periodic solution. Qualitative rate-of-growth information may also be obtained from this method. In implementing Birkhoff's "normalization" process, one carries out only a finite number of steps, of course. Accordingly, the solution obtained is that for a differential system approximating

*Known also as the reduced three-body problem.

the original Hamiltonian system. Even so, using contemporary symbolic manipulation techniques and high speed digital computers, it is practical to retain enough nonlinear terms in the normalization process so that accurate solutions are obtained, valid over long time intervals. Thus the Birkhoff normalization technique is a means to analyze, in a quantitative way, finite-time stability properties of nonlinear dynamical systems, without recourse to numerical integration and its attendant errors.

The Birkhoff normalization process is applicable to:

Study of trajectories near an equilibrium point of a conservative Hamiltonian system.

Study of trajectories near an equilibrium point of a nonconservative Hamiltonian system having a periodic Hamiltonian function.

In a conservative Hamiltonian system, study of trajectories near a periodic trajectory which lie on a surface having the same energy as the given periodic trajectory.

The planar circular restricted three-body problem is a conservative Hamiltonian system with equilibrium points, and we have applied the Birkhoff normalization process to that system [9] in an earlier study.

This report describes the application of the Birkhoff normalization process to a nonconservative Hamiltonian system with a periodic Hamiltonian function and equilibrium points. In the next two sections, we will describe in detail the implementation of the Birkhoff normalization process for an arbitrary Hamiltonian function of this type. We will then consider specifically the planar elliptical restricted three-body problem, and its triangular libration point L_4.

II. NORMALIZATION PROCEDURES

The normalization procedure is applied to time-dependent Hamiltonians expanded about an equilibrium point which is also the origin of coordinates. In this form, the Hamiltonian is

$$H = H_o(t) + H_2(x,y,t) + \ldots + H_n(x,y,t) + \ldots , \qquad (II.1)$$

where H_n is a homogeneous polynomial of degree n in the conjugate vector x,y, with coefficients having period 2π in t. As $H_o(t)$ plays no role in the analysis, we will not consider it. A Hamiltonian is considered normalized to a finite order when it has the form

$$H(x,y,t) = H_a\left[(x_1^2 + y_1^2) , (x_2^2 + y_2^2) \ldots\right] + H_b[x,y,t] . \qquad (II.2)$$

For small values of the state variable (x,y) the higher homogeniety portion, H_b can be dropped resulting in

$$H \approx H_a$$

where the exact solution for H_a exists and is an harmonic oscillator for each pair of conjugate variables. The chief requirement of the normalization procedure is to find canonical coordinate transformations that will take the original Hamiltonian (II.1) and put it in the form of (II.2). These required transformations fall into two separate categories, where the first normalizes the Hamiltonian to second order [10]

$$H(x,y,t) = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) + \ldots + H_b(x,y,t) \qquad (II.3)$$

and the second systematically normalizes the Hamiltonian to higher orders by a repeated application of Birkhoff's normalization procedure [8].

II.1 2nd Order Normalization

Each of these categories will be discussed separately starting with second order normalization. Let us represent a linear Hamiltonian system of dimension $2n$ with a periodic time

dependency by

$$\dot{w} = A(t)w, \quad A(t + 2\pi) = A(t) \quad . \tag{II.4}$$

Then there exists a canonical change of variables

$$w = P(t)z, \quad P(t + 2\pi) = P(t) \quad , \tag{II.5}$$

such that

$$\dot{z} = Bz \quad , \tag{II.6}$$

where B is a constant matrix, if the characteristic exponents associated with (II.4) are distinct.

By Floquet theory, [10], the fundamental matrix $\Phi(t)$ of (II.4) with $\Phi(0) = I$, can be represented as

$$\Phi(t) = P(t) \exp(Bt) \quad , \tag{II.7}$$

where

$$B = \frac{1}{2\pi} \log_e \Phi(2\pi)$$

and

$$P(t + 2\pi) = P(t) \quad .$$

The change of variables (II.5) gives (II.6). As the characteristic exponents of our system are distinct, it can be shown, [11], that there exists an S such that

$$S^{-1}BS = D \quad ,$$

where D is diagonal. Furthermore, as we have a Hamiltonian system, the eigenvalues of D are $\omega_1, \ldots, \omega_n, -\omega_1, \ldots, -\omega_n$. By applying the linear, canonical change of variables

$$w = P(t)Sr \quad ,$$

where

$$r' = (p_1, p_2, q_1, q_2) \quad \text{and} \quad w' = (x, y) \quad ,$$

it follows readily that the equations defined by H take the form

$$\dot{r} = Dr + \dots \quad . \tag{II.8}$$

Thus in the new variables, we have, by a linear transformation to real form

$$\hat{H} = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2) + \dots \quad .$$

Equations (II.8) are now integrable if terms of degree two and higher are neglected. This linear approximation is good over some fixed finite time interval if we restrict ourselves to a sufficiently small neighborhood of the equilibrium point.

II.2 Higher Order Normalization

The next step in the analysis is to consider the higher order normalization procedure. We assume that H has been normalized up to degree s - 1 and introduce a canonical transformation from the old variables (x,y) to new variables (ξ,η) generated by a generating function $W^{(s)}(x_1,\dots,x_n,\eta_1,\dots,\eta_n)$, a homogeneous polynomial of degree s. The transformation is given (implicitly) by

$$\begin{aligned} \xi_i &= x_i + \frac{\partial W^{(s)}}{\partial \eta_i} \quad , \\ y_i &= \eta_i + \frac{\partial W^{(s)}}{\partial x_i} \quad . \end{aligned} \tag{II.9}$$

The idea is to choose $W^{(s)}$ (that is, the coefficients of the polynomial) in such a way that as many s-th order terms as possible are eliminated in the new Hamiltonian. If $\Gamma(\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n)$ is the new Hamiltonian, we have

$$H(x_1,\dots x_n,\eta_1 + \frac{\partial W^{(s)}}{\partial x_1},\dots,\eta_n + \frac{\partial W^{(s)}}{\partial x_n}) = \Gamma(x_1 + \frac{\partial W^{(s)}}{\partial \eta_1},\dots,x_n + \frac{\partial W^{(s)}}{\partial \eta_n},\eta_1,\dots,\eta_n) \quad .$$

We expand both sides in Taylor series and find

$$H(x_1,\dots x_n,\eta_1\dots\eta_n) + \sum_{i=1}^{n} \frac{\partial H}{\partial \eta_i}\frac{\partial W^{(s)}}{\partial x_i} + \frac{1}{2!}\sum_{\substack{i=1\\ j=1}}^{n} \frac{\partial^2 H}{\partial \eta_i\,\partial \eta_j}\frac{\partial W^{(s)}}{\partial x_i}\frac{\partial W^{(s)}}{\partial x_j} + \dots.$$

$$= \Gamma(x_1,\dots x_n,\eta_1\dots\eta_n) + \sum_{i=1}^{n} \frac{\partial \Gamma}{\partial x_i}\frac{\partial W^{(s)}}{\partial \eta_i} + \frac{1}{2!}\sum_{\substack{i=1\\ j=1}}^{n} \frac{\partial^2 \Gamma}{\partial x_i\,\partial x_j}\frac{\partial W^{(s)}}{\partial \eta_i}\frac{\partial W^{(s)}}{\partial \eta_j} + \dots \quad . \tag{II.10}$$

We now write H and Γ as sums of homogeneous polynomials:

$$H = \sum_{n=2}^{\infty} H^{(n)} \quad , \quad \Gamma = \sum_{n=2}^{\infty} \Gamma^{(n)} \quad .$$

We equate terms of like degree in (II.10) and find

$$H^{(i)}(x_1,\dots,x_n,\eta_1,\dots,\eta_n) = \Gamma^{(i)}(x_1,\dots,x_n,\eta_1,\dots,\eta_n) \quad , \quad i = 1,2,\dots s-1 \ ,$$

so that the new Hamiltonian is normalized up to order $s-1$ since it was assumed that H was. Equating terms of degree s in (II.10), we have

$$H^{(s)} + \sum_{i=1}^{n} \frac{\partial H^{(a)}}{\partial \eta_i}\frac{\partial W^{(s)}}{\partial x_i} = \Gamma^{(s)} + \sum_{i=1}^{n} \frac{\partial \Gamma^{(a)}}{\partial x_i}\frac{\partial W^{(s)}}{\partial \eta_i} \quad .$$

In view of (II.3), this can be written

$$\sum_{i=1}^{n} \omega_i \left(\eta_i \frac{\partial W^{(s)}}{\partial x_i} - x_i \frac{\partial W^{(s)}}{\partial \eta_i} \right) = \Gamma^{(s)} - H^{(s)} \quad . \tag{II.11}$$

When coefficients of like terms are compared in the last equation, we obtain a system of linear equations in the coefficients of $W^{(s)}$ with unspecified non-homogeneous terms since $\Gamma^{(s)}$ is not known. It turns out that for s odd, the matrix of this system is non-singular if $\sum_{i=1}^{n} \omega_i k_i \neq 0$ for integers k with $0 < \Sigma |k_i| \leq N$, so that the system can be solved

with an arbitrary non-homogeneous term, i.e. we can find a $W^{(s)}$ so that (II.11) is satisfied with $\Gamma^{(s)} \equiv o$. For even s the matrix does turn out to be singular. In this case, we can determine $W^{(s)}$ so that (II.11) is satisfied with $\Gamma^{(s)}$ a polynomial of degree $s/2$ in the variables $(x_i^2 + \eta_i^2)$ with known coefficients. Thus the new Hamiltonian is normalized up to order s. The higher order terms in Γ are then found by coefficient comparison in (II.10), since now $W^{(s)}$ is known.

The following flow diagram of Figure 1 illustrates the process used for higher order normalization.

HIGHER ORDER NORMALIZATION FLOW DIAGRAM

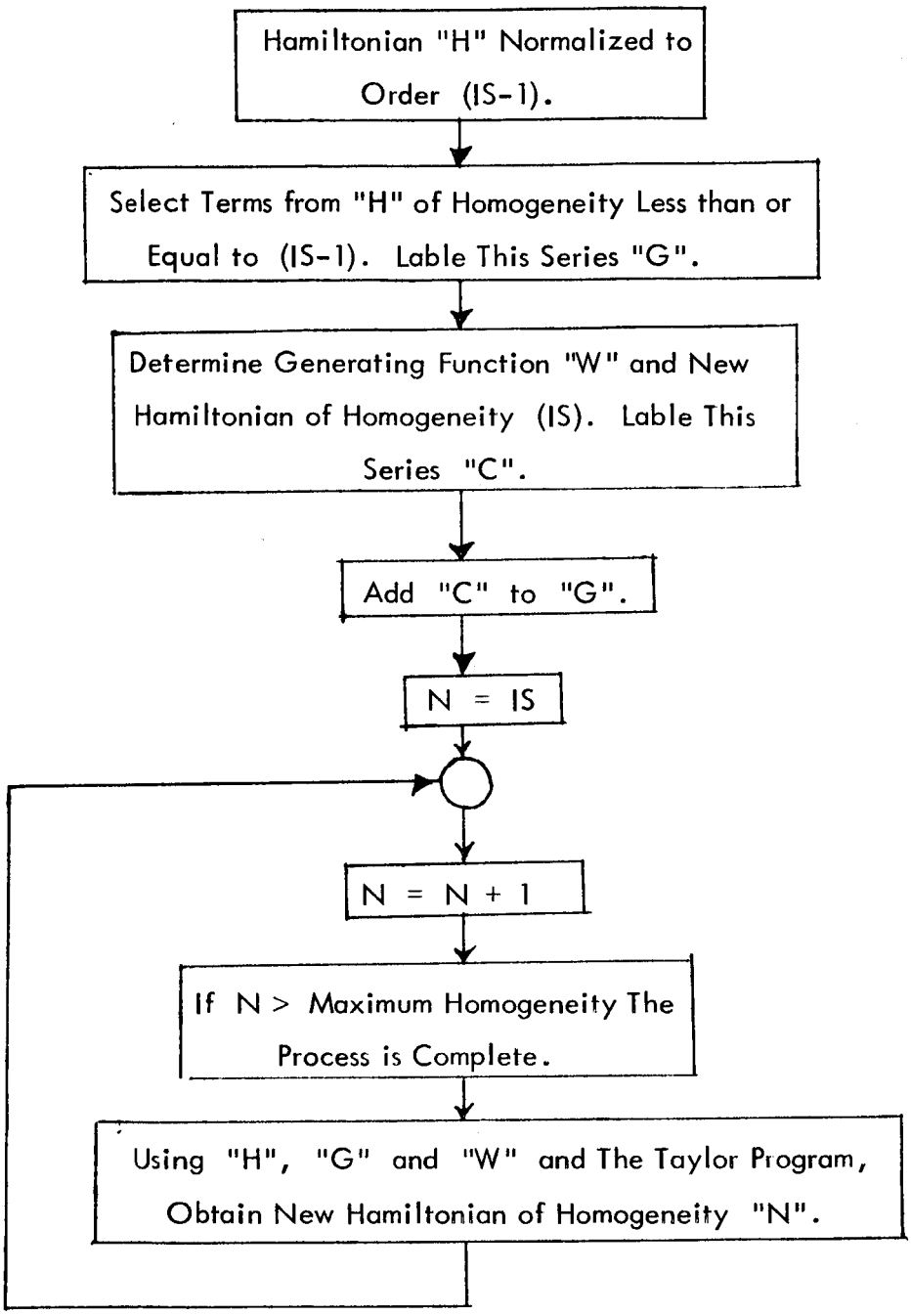

FIGURE 1

III. NUMERICAL IMPLEMENTATION

In general, the Hamiltonian in the neighborhood of a periodic solution may be represented in the form

$$H(x,y,t) = \sum_{\nu_1 + \ldots + \nu_{2n} = 2}^{\infty} a_{\nu_1, \ldots, \nu_{2n}} \; x_1^{\nu_1} \ldots x_n^{\nu_n} \; y_1^{\nu_{n+1}} \ldots y_n^{\nu_{2n}}$$

where the coefficients are periodic functions of time. The first step in the normalization process is to reduce the periodic matrix of the linearized system derived from the Hamiltonian to a constant matrix in real normal form. The resulting Hamiltonian then has the form

$$H(x,y,t) = \sum_{i=1}^{n} \frac{\omega_i}{2} (x_i^2 + y_i^2) + \sum_{\nu_1 + \ldots + \nu_{2n} = 3}^{\infty} a_{\nu_1, \ldots, \nu_{2n}} \; x_1^{\nu_1} \ldots x_n^{\nu_n} \; y_1^{\nu_{n+1}} \ldots y_n^{\nu_{2n}}$$

which is suitable for higher order normalization.

The following programs are used to automate the entire process for both the second order and higher order normalizations.

III. 1 Symbolic Technique

Throughout the programs a special technique is used for the handling of series terms or matrix elements which are series. Three arrays(NC1, C1, IC1) are used to completely specify either condition.

1) NC1 - Used to identify the number of terms in the series or in the matrix element.
2) C1 - Contains the coefficients of each term in the series.
3) IC1 - Logical definition of each term where the description variable is in packed form by bit designation. This term may appear in either of two forms:

1. 7 exponential and one time term, i.e.:

$$c\, t^{KT} \prod_{k=1}^{7} \exp i(-1)^{J_k} K_k t \ , \qquad i = \sqrt{-1} \ ,$$

which is expressed in packed form as

35	34	33	...	28	27-23	22-20	...	7-5	4-1
I	J_1	J_2	...	J_7	K_1	K_2	...	K_7	KT

where

$$I = \begin{cases} 0, \text{ c real} \\ 1, \text{ c imaginary} . \end{cases}$$

KT = exponent of the time term.

$$J_k = \begin{cases} 0, \text{ for positive exponential time dependence} \\ 1, \text{ for negative exponential time dependence}, \end{cases}$$

$k = 1, \ldots, 7.$

K_k = positive integer, less than or equal to 31,

$k = 1, \ldots, 7.$

2. one exponential and six spatial terms, i.e.

$$c\left(\prod_{k=1}^{6} x_k^{L_k}\right) \exp i(-1)^J K t, \qquad i=\sqrt{-1},$$

which is expressed in packed form as

35	34	33-25	24-21	20-17	16-13	12-9	8-5	4-1
I	J	K	L_1	L_2	L_3	L_4	L_5	L_6

where

$$I=\begin{cases}0, \text{ c real}\\ 1, \text{ c imaginary}\end{cases}$$

$$J=\begin{cases}0, \text{ for positive exponential time dependence}\\ 1, \text{ for negative exponential time dependence.}\end{cases}$$

K = positive integer, less than or equal to 63.

L_k = exponent of space coordinate x_k, k = 1, . . . , 6.

III. 2 Program Descriptions

Program No. 1

This program determines the system of linear differential equations associated with the Hamiltonian as well as computing its matrix of eigenvectors.

The functions performed by the program are as follows:

1. Select the second order terms from the original Hamiltonian and set up linear differential equations in matrix form:

$$\dot{X} = A(t)\, X$$

or

$$\dot{x}_i = (-1)^k \frac{\partial H}{\partial x_k} = \sum_{j=1}^{n} a_{kj}\, x_j \quad ,$$

$$k = \begin{cases} i+1, & i \text{ odd} \\ i-1, & i \text{ even} \end{cases}, \quad i = 1, \ldots, n \ ,$$

where n is the number of dependent variables, $n \leq 6$.

2. The coefficient array $A(t)$ is separated into discrete exponential orders of time. $A(t)$ is a matrix containing four exponential and no spatial terms.

$$A(t) = A(0) + A(1) + \ldots + A(N)$$

3. C and C^{-1} are found where the transformation of coordinates is

$$X = C\, Y \quad ,$$

where Y represents the coordinates in which $A(0)$ is diagonal, and C is the

Program No. 1 (continued)

matrix of eigenvectors of A(0) where C^{-1} A(0) C diagonalizes A(0).

4. The matrix $B = C^{-1} A(t) C$ is also computed.

Supporting Subroutines

1. AMAT
2. COMIN
3. COMOP
4. CONV
5. DCUBIC
6. DQRTIC
7. EIGEN
8. EIGVAL
9. EIGVEC
10. IDENT
11. IDENTI
12. LINMAT
13. MATIO
14. MATMUL
15. MATOP1
16. MATPRT
17. PROOT
18. SERDET
19. SERIO
20. SEROP
21. SEROP1
22. SERPRT

Input

1. NIP(I), I = 1,72 - Print control where I represents the subroutine number.
2. NH - Number of terms in the time-dependent Hamiltonian series.
3. ISET(2) = $\begin{cases} 0, & \text{Series is in the form of 1 exponential and 6 spatial terms,} \\ 1, & \text{Series is in the form of 7 exponential and 1 time term.} \end{cases}$
4. H - Coefficients of the Hamiltonian series.
5. IH - Logical definition of a term in the Hamiltonian series. The following terms are used for this description:

   a) I = $\begin{cases} 0, & \text{coefficient c is real} \\ 1, & \text{coefficient c is imaginary} \end{cases}$

   b) J = $\begin{cases} 0, & \text{for positive exponential time dependence} \\ 1, & \text{for negative exponential time dependence} \end{cases}$

   c) K - Integer multiplying the time in the exponential.

   d) L_k - Exponent of the kth spatial coordinate, k = 1, . . . , 6.

Printed Output

1. NH
   H — Input Hamiltonian series.
   IH
2. XLAM — Eigenvalues of the A(0) array.
3. NAM
   AM — A(t) array. Matrix type is 7 exponential and one time term.
   IAM
4. NBM
   BM — Constant B array. Matrix type is 7 exponential and one time term.
   IBM

5. NCM — Eigenvector array C. Matrix type is 7 exponential and one
   CM — time term.
   ICM
6. NCIM
   CIM — Inverse of the eigenvector matrix.
   ICIM

Punched Output

1. NAM
   AM — A(t) array. Matrix type is 7 exponential and one time term.
   IAM
2. NBM
   BM — Constant B array. Matrix type is 7 exponential and one time term.
   IBM
3. NCM — Eigenvector array C. Matrix type is 7 exponential and one time
   CM — term. Each element is in complex form.
   ICM
4. NCIM
   CIM — Inverse of the eigenvector matrix.
   ICIM

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PROGRAM No. 1	PUNCHING INSTRUCTIONS	PAGE OF
	GRAPHIC	CARD ELECTRO NUMBER
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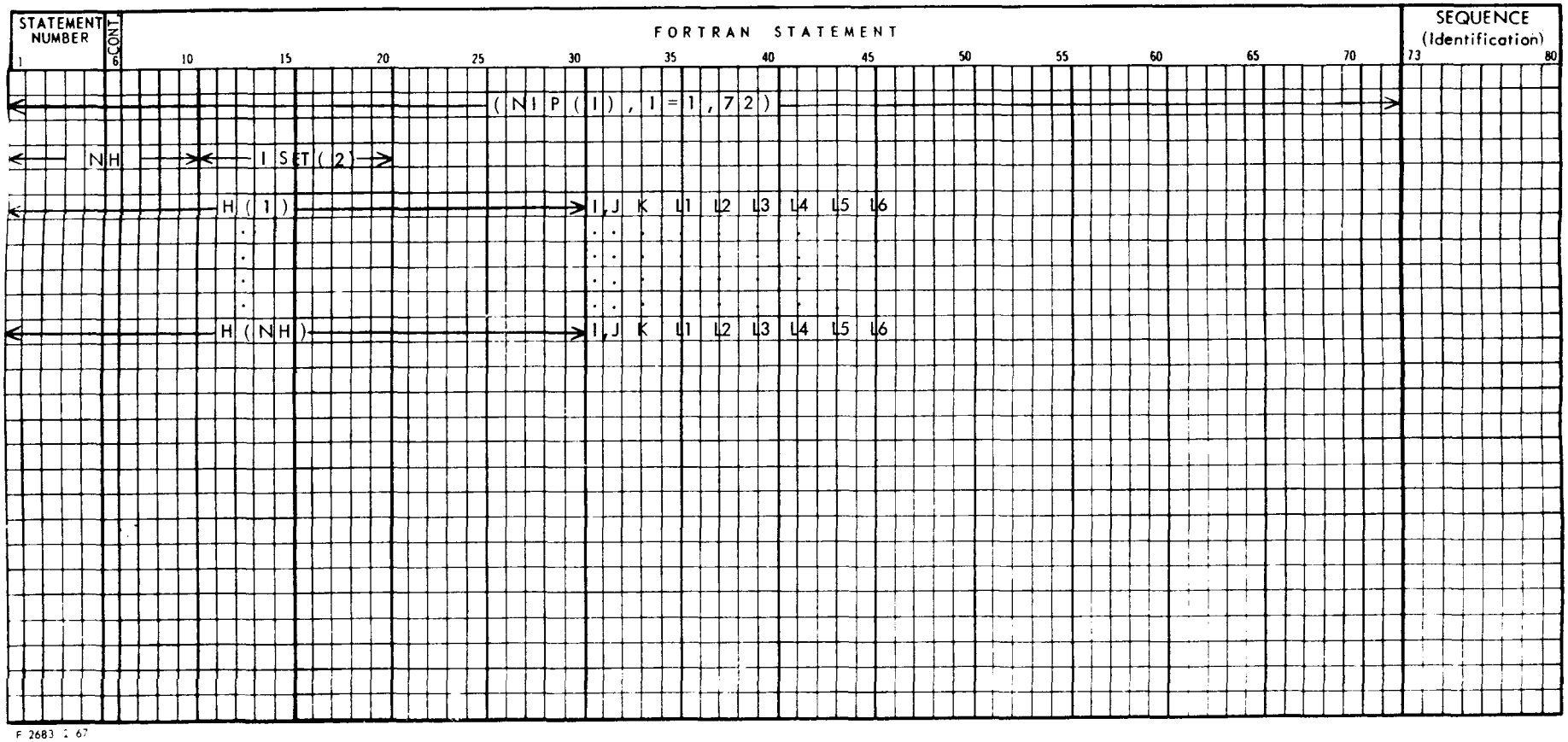

F 2683 : 67

Program No. 2

This program solves the linear system resulting in the characteristic matrix $\Phi(t)$.

The linear differential equation

$$\dot{Y} = B(t)\,Y\ ,$$

where $B(0)$ is a constant diagonal matrix with imaginary diagonal elements, is solved in symbolic form, yielding the matrix (YM). Each element in B can be expanded in a Fourier series, so that

$$B = B(0) + B(1) + B(2)\ \ldots.$$

The following recursion technique is used to obtain the solution:

$$\dot{Y}(1) = B(0)\ Y(1)$$
$$\dot{Y}(2) = B(0)\ Y(2) + B(1)\ Y(1)$$
$$\dot{Y}(3) = B(0)\ Y(3) + B(1)\ Y(2) + B(2)\,Y(1)$$
$$\vdots$$
$$\dot{Y}(NORD+1) = B(0)\ Y(NORD+1) + B(1)Y(NORD) + \ldots,$$

where NORD = number of recursion steps permitted after solution of the first linear equation. The i th column of (YM) represents the i th independent solution.

A general solution for Y is

$$Y = (YM)\,D,$$

where D is a constant matrix.

The solution for X in the original equation,

$$\dot{X} = A(t)\,X\ ,$$

is

$$X = CY = C(YM)\,D.$$

Program No. 2 (continued)

At time zero, (YM) is the identity matrix and it is desirable to have X become the identity matrix. Accordingly, we set $D = C^{-1}$. Then $\Phi(t) = C(YM)C^{-1}$.

As an example of the results obtained by this procedure the following two-dimensional problem is solved up to first order.

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} 2.35i & .04\,i\,e^{+it} \\ +\ .05\,e^{+it} & \\ .07\,e^{-it} & -2.35i \\ & +\ .08e^{-it} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

The solution for the characteristic matrix $\Phi(t)$ is;

$$\Phi(t) = \begin{bmatrix} e^{2.34it} - .05\,i\,e^{+it} & -.0108\,e^{+it}\,e^{-2.35it} \\ +.05\,i\,e^{+it}\,e^{+2.5it} & +.0108\,e^{+2.35it} \\ -.0189\,i\,e^{-it}\,e^{+2.35it} & 1.08\,e^{-2.35it} \\ +.0189\,i\,e^{-2.35it} & -.08\,e^{-it}\,e^{-2.35it} \end{bmatrix}$$

Each column of $\Phi(t)$ represents an independent solution for the Y vector.

Due to the excessive storage requirements necessary for the recursion process, various arrays were stored on the Univac FH880 drum in the form of simulated scratch tapes.

Program No. 2

Supporting Subroutines

1. CONV
2. IDENT
3. IDENT1
4. MATIO
5. MATOP1
6. MATPRT
7. SDIF1
8. SEROP
9. SEROP1
10. SERPRT
11. SINT1
12. YMAT

Input

1. NIP(I) I = 1, 72 — Print control where I represents the subroutine number.
2. NBM, BM, IBM — B array from Program No. 1.
3. NCM, CM, ICM — Eigenvector array C from Program No. 1.
4. NCIM, CIM, ICIM — C^{-1} from Program No. 1.
5. NORD — Order to which the recursive solution for YM in subroutine YMAT is to be computed.

Printed Output

1. NBM, BM, IBM — Input matrix from Program No. 1.
2. NCM, CM, ICM — Input eigenvector matrix from Program No. 1.

Program No. 2

Printed Output (continued)

3. NCIM
   CIM
   ICIM

   Input inverse of the eigenvector matrix from Program No. 1.

4. NYM
   YM
   IYM

   YM Array identified in previous description.

5. NPHIM
   PHIM
   IPHIM

   Characteristic matrix $\Phi(t)$.

Punched Output

1. XLAM (I) , I = 1, 13

   Diagonal elements of the BM matrix.

2. NPHIM
   PHIM
   IPHIM

   $\Phi(t)$. Matrix type is 7 - exponential and one time term.

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PROGRAM No. 2		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER (1–5)	CONT (6)	FORTRAN STATEMENT (7–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
		NBM,BM,IBM FROM PROGRAM NO.1	
		NCM,CM,ICM FROM PROGRAM NO.1	
		NCIM,CIM,ICIM FROM PROGRAM NO.1	
NORD			

F 2683 2 67

Program No. 3

By Floquet theory the characteristic matrix $\Phi(t)$ is defined as:

$$\Phi(t) = P(t)\ e^{Bt}$$

where $P(t)$ is the periodic coordinate transformation matrix and B is a real constant matrix defined as

$$B = \frac{1}{2\pi}\ \log_e \Phi(2\pi)\ .$$

The following mathematical procedure is used to evaluate the logarithm of a real matrix:

a) Find the eigenvector matrix C of the $\Phi(2\pi)$ matrix as well as the inverse of C, denoted by CI.

b) Form the complex diagonal matrix $D = CI \cdot A \cdot C$.

c) Take the logarithm of the principal values of the diagonal elements in the D matrix. The elements of D are in the form $x + iy$ which is converted to $R\,e^{i\theta}$, $0 \le \theta \le 2\pi$.

$$G = \log_e R\ e^{i\theta} = \log_e R + i\theta\ .$$

d) The desired matrix is $C \cdot G \cdot CI$, i.e.

$$B = \frac{1}{2\pi}\ C\left[\log_e (C^{-1}\ \Phi(2\pi)\ C)\right] C^{-1}.$$

Now $P(t) = \Phi(t)\ e^{-Bt}$.

Program No. 3 evaluates

$$E(t) = e^{-Bt} = S\left[e^{-t(SI \cdot B \cdot S)}\right] SI \quad ,$$

where S is the eigenvector matrix of B , and SI denotes the inverse of S .

Supporting Subroutines

1. COMIN
2. COMOP
3. CONV
4. DCUBIC
5. DQRTIC
6. EIGEN
7. EIGVAL
8. EIGVEC
9. EMAT
10. IDENT
11. IDENT1
12. LINMAT
13. MATIO
14. MATLOG
15. MATMUL
16. MATOP1
17. MATPRT
18. PROOT
19. SERDET
20. SERNUM
21. SEROP
22. SEROP1
23. SERPRT

Input

1. NIP(I), I = 1, 72	Print control where I represents the subrountine no.
2. XLAM(I) , I = 1, 3	Eigenvalues of $\Phi(t)$ matrix from Program No. 2.
3. NPHIM PHIM IPHIM	$\Phi(t)$ array from Program No. 2.

Printed Output

1. XLAM(I) , I = 1, 3	Eigenvalues of characteristic matrix $\Phi(t)$ from Program No. 2.
2. NPHIM PHIM IPHIM	$\Phi(t)$ matrix from Program No. 2.
3. XLAME(I) , I = 1, 3	Eigenvalues of exponential array E(t) .
4. NEM EM IEM	E(t) array .
5. BB	Real constant matrix B described in program description.
6. BLAM	Eigenvalues of the constant B matrix.
7. CB	Eigenvector of the constant B matrix.

Punched Output

XLAM XLAME EM BLAM CB	Described in printed output.

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PROGRAM No. 3		PUNCHING INSTRUCTIONS	PAGE OF
		GRAPHIC	CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH	

STATEMENT NUMBER (1–5)	CONT (6)	FORTRAN STATEMENT (7–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
		XLAM(1) XLAM(2) XLAM(3)	
		NPHIM,PHIM,IPHIM FROM PROGRAM NO. 2	

F 2683 2 67

Program No. 4

This program performs the multiplication of $\Phi(t)$ and $E(t)$ to produce the periodic matrix $P(t)$, and expands the matrix $P(t)$ in a Fourier series. The resultant series is periodic with period 2π.

Supporting Subroutines

1. CONV
2. IDENT
3. IDENT1
4. MATIO
5. MATOP1
6. MATPRT
7. SDIF1
8. SEREX1
9. SERNUM
10. SEROP
11. SEROP1
12. SERPRT
13. SINT1

Input

1. NIP(I), I = 1, 72	Print control where I represents the subroutine number.
2. XLAM(J), J = 1,3	Eigenvalues of the $\Phi(t)$ array from Program No. 3.
3. XLAME(J), J = 1,3	Eigenvalues of the E(t) array from Program No. 3.
4. NPHIM PHIM IPHIM	$\Phi(t)$ array from Program No. 2. Matrix type is 7 exponential and one time term.
5. NEM EM IEM	E(t) array from Program No. 3. Matrix type is 7 exponential and one time term.

Program No. 4

Input (continued)

6. NEX	Degree of expansion maintained in generation of the periodic matrix P(t) .

Printed Output

1. NEX	Degree maintained in the periodic matrix expansion.
2. XLAM	Eigenvalues of the Φ (t) matrix.
3. NPHIM PHIM IPHIM	Characteristic matrix Φ(t) from Program No.2.
4. XLAME	Eigenvalues of the exponential array E(t) .
5. NEM EM IEM	Exponential array E(t) from Program No. 3.
6. NPM PM IPM	Periodic matrix P(t) .

Punched Output

1. NPM PM IPM	Periodic matrix P(t). Matrix type is one exponential and 6 spatial terms.

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PROGRAM No. 4		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER (1–5)	CONT (6)	FORTRAN STATEMENT (7–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
		XLAM FROM PROGRAM NO. 3	
		XLAME FROM PROGRAM NO. 3	
		NPHIM,PHIM,IPHIM FROM PROGRAM NO. 2	
		NEM,EM,IEM FROM PROGRAM NO. 3	
		NEX	

Program No. 5

This program determines the transformation necessary to put the constant portion of the second order terms in real normal form. The result of this program is a matrix T (t) representing the total transformation from the original form to real normal form.

The detailed functions that are performed are:

1. The matrix DEL which preserves the canonical property is obtained.
2. The matrix RNF that transforms a complex normal Hamiltonian into real normal form is obtained.
3. The resultant transformation T (t) that normalizes the Hamiltonian to second order is found:

$$T(t) = P(t) \cdot CB \cdot DEL \cdot RNF$$

Supporting Subroutines

1.	CANMAT	7.	MATMUL
2.	COMOP	8.	MATOP
3.	CONV	9.	MATPRT
4.	IDENT	10.	SEROP
5.	IDENTI	11.	SERPRT
6.	MATIO		

Input

1.	NIP(I), I = 1,72	Print control where I represents the subroutine number.
2.	NPM PM IPM	Periodic transformation matrix P (t) from Program No. 4.
3.	CB	Matrix of eigenvectors of the B array from Program No. 3.

Printed Output

1.	NPM PM IPM	Input periodic matrix P (t) from Program No. 4.
2.	DEL	Matrix which preserves the canonical property.
3.	RNF	Matrix which transforms a complex normal Hamiltonian into real normal form.

Printed Output (cont'd)

4. NTM TM ITM	Resultant transformation matrix T (t).

Punched Output

NTM TM ITM	Total transformation matrix. Matrix type is one exponential and 6 spatial terms.

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FORTRAN CODING FORM

PROGRAM No. 5		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER (1–5)	CONT (6)	FORTRAN STATEMENT (10–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
		NPM,PM,IPM FROM PROGRAM NO. 4	
		CB FROM PROGRAM NO. 3	

F 2683 2 67

Program No. 6

The functions performed by this program are as follows:

1. The Z series is obtained where the coordinate transformation that removes time dependence from second order terms is

$$X = T(t)\, Z\ .$$

2. The above linear transformation is substituted into the original Hamiltonian and yields a Hamiltonian normalized to second order in real normal form.
3. The old variables X are obtained in terms of the new variables Z and the new variables Z are obtained in terms of the old variables X.

Supporting Subroutines

1. COMIN	9. MATMUL
2. COMOP	10. MATOP
3. CONV	11. MATPRT
4. IDENT	12. SERIO
5. IDENT1	13. SERNUM
6. LINMAT	14. SEROP
7. LINTRA	15. SERPRT
8. MATIO	16. STRAN

Input

1. NIP(I), I = 1,72	Print control where I represents the subroutine number.
2. ISET (4)	Maximum time order carried.
3. NH H IH	Original Hamiltonian.
4. NTM TM ITM	Linear time dependent coordinate transformation from Program No. 5.

Printed Output

1. NH H IH	Initial Hamiltonian series.
2. NTM TM ITM	Total transformation matrix T (t) from Program No. 6.

Printed Output (cont'd)

3. NG G IG	New Hamiltonian normalized to second order.
4. NG G IG	New Hamiltonian after trivial second order terms have been removed.
5. NH H IH	Old coordinates X expressed in terms of the new coordinates Z.
6. NH H IH	New coordinates Z expressed in terms of the old coordinates X.

Punched Output

1. NG G IG	New Hamiltonian normalized to second order.
2. X	Old coordinates in terms of the new coordinates Z.
3. Z	New coordinates in terms of the old coordinates.

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FORTRAN CODING FORM

PROGRAM No. 6		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER (1–5)	CONT (6)	FORTRAN STATEMENT (10–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
		ISET(4)	
		NH,H,IH - SAME INPUT AS PROGRAM NO. 1	
		NTM,TM,ITM FROM PROGRAM NO. 5	

F 2683 2 67

Program No. 7

This program computes the generating function necessary to normalize the Hamiltonian to the next higher order. A basic requirement of the normalization process is the expansion of polynomials by Taylor series. In particular, consider the expansion about (x, y) of

$$A\left(x, y + \frac{\partial B}{\partial x}\right)$$

where "A" and "B" are series and "x" and "y" are portions of the total coordinate state vector. This series is given by

$$\sum_{n=1}^{m} \sum_{l(1)=1}^{ndif} \cdots \sum_{l(n)=1}^{ndif} \frac{\partial^n A}{\partial[y_{l(1)}] \cdots \partial[y_{l(n)}]} \frac{\partial B}{\partial[x_{l(1)}]} \cdots \frac{\partial B}{\partial[x_{l(n)}]}$$

where "m" is the order of the expansion and "ndif" is the number of components of the "x" vector. An additional constraint on the formulation of the expansion is that there is a need for only a portion of the total expansion, namely one homogeneity level, so this must be selected without the computation of the entire result.

As an example consider,

$$\begin{aligned} A = \quad & + 2. \quad e^{0t} \; x_1^2 \; x_2^0 \; x_3^0 \; x_4^0 \\ & + 3. \quad e^{0t} \; x_1^2 \; x_2^1 \; x_3^0 \; x_4^0 \\ & + 4. \quad e^{0t} \; x_1^1 \; x_2^3 \; x_3^0 \; x_4^0 \\ & + 5. \quad e^{0t} \; x_1^3 \; x_2^2 \; x_3^0 \; x_4^0 \end{aligned} ,$$

$$B = \begin{array}{llllll} +3. & e^{0t} & x_1^1 & x_2^2 & x_3^0 & x_4^0 \\ +10. & e^{0t} & x_1^2 & x_2^1 & x_3^0 & x_4^0 \end{array},$$

with

$$x_{1(1)} = y_{1(1)} = x_1$$
$$x_{1(2)} = y_{1(2)} = x_2$$
$$x_{1(3)} = y_{1(3)} = x_3$$

The resulting series of homogeneity 4 is

$$C = \begin{array}{llllll} +42. & e^{0t} & x_1^1 & x_2^3 & x_3^0 & x_4^0 \\ +20. & e^{0t} & x_1^2 & x_2^2 & x_3^0 & x_4^0 \\ +18. & e^{0t} & x_1^3 & x_2^1 & x_3^0 & x_4^0 \\ +3. & e^{0t} & x_1^4 & x_2^0 & x_3^0 & x_4^0 \\ +18. & e^{0t} & x_1^0 & x_2^4 & x_3^0 & x_4^0 \end{array}.$$

The new Hamiltonian is also evaluated in the new coordinates, i.e. a basic series coordinate transformation.

Consider as an example, the series

$$H(x_1, x_2, x_3, x_4) = \begin{array}{llllll} +2i & e^{-3it} & x_1^2 & x_2^0 & x_3^0 & x_4^1 \\ +3 & e^{+3it} & x_1^1 & x_2^0 & x_3^0 & x_4^2 \end{array}$$

and consider the transformation defined by

$$
\begin{aligned}
x_1 &= +3i\ e^{-3it}\ y_1^1 y_2^0 y_3^0 y_4^0 \\
&\quad +5\ e^{2it}\ y_1^2 y_2^0 y_3^0 y_4^0 \\
x_2 &= +1\ e^{0it}\ y_1^0 y_2^0 y_3^0 y_4^0 \\
x_3 &= +3i\ e^{-3it}\ y_1^0 y_2^0 y_3^1 y_4^0 \\
&\quad +5\ e^{2it}\ y_1^0 y_2^0 y_3^2 y_4^0 \\
x_4 &= +3i\ e^{-3it}\ y_1^0 y_2^0 y_3^0 y_4^1 \\
&\quad +5\ e^{2it}\ y_1^0 y_2^0 y_3^0 y_4^2 .
\end{aligned}
$$

In the new coordinates,

$$
\begin{aligned}
\tilde{H}(y_1, y_2, y_3, y_4) = {} & +54\ e^{-12it}\ y_1^2 y_2^0 y_3^0 y_4^1 \\
& -90i\ e^{-7it}\ y_1^2 y_2^0 y_3^0 y_4^2 \\
& -180i\ e^{-7it}\ y_1^3 y_2^0 y_3^0 y_4^1 \\
& -300\ e^{-2it}\ y_1^3 y_2^0 y_3^0 y_4^2 \\
& -150\ e^{-2it}\ y_1^4 y_2^0 y_3^0 y_4^1 \\
& +250i\ e^{+3it}\ y_1^4 y_2^0 y_3^0 y_4^2 \\
& -81i\ e^{-6it}\ y_1^1 y_2^0 y_3^0 y_4^2 \\
& -270\ e^{-1it}\ y_1^1 y_2^0 y_3^0 y_4^3 \\
& +225i\ e^{+4it}\ y_1^1 y_2^0 y_3^0 y_4^4 \\
& -135\ e^{-1it}\ y_1^2 y_2^0 y_3^0 y_4^2 \\
& +450i\ e^{+4it}\ y_1^2 y_2^0 y_3^0 y_4^3 \\
& +375\ e^{+9it}\ y_1^2 y_2^0 y_3^0 y_4^4 .
\end{aligned}
$$

This process is repeated until the desired level of normalization is achieved. The time dependent Hamiltonian used for input must be normalized to second order.

Supporting Subroutines

1. GEN
2. IDENT
3. IDENT1
4. NORM1
5. SERIO
6. SEROP
7. SERPRT
8. STRAN
9. TAY1
10. TAY2

Input

1.	NIP(I), I = 1,72	Print control where I represents the subroutine number.
2.	INORM	Normalization level of the initial Hamiltonian.
3.	MAXHOM	Maximum order to which the Hamiltonian will be normalized.
4.	NH	
	H	Initial Hamiltonian normalized to second order or higher.
	IH	

Printed Output

1.	INORM	Normalization level of input Hamiltonian, usually 2.
2.	MAXHOM	Maximum order to which the Hamiltonian will be normalized.
3.	OMEGA	Constants multiplying the second order terms in the input Hamiltonian.
4.	NH	
	H	Input Hamiltonian.
	IH	
5.	NW	
	W	Generating function for Hamiltonian normalized to one higher order.
	IW	
6.	NG	
	G	Hamiltonian normalized to higher order.
	IG	

Punched Output

1. NW

   W

   IW — Generating function for the Hamiltonian normalized to one higher order.

2. NG

   G

   IG — Hamiltonian normalized to one higher order.

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PROGRAM No. 7		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER (1–5)	CONT. (6)	FORTRAN STATEMENT (7–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
		INORM	
		MAXHOM = ISET(4)	
		NH,H,IH FROM PROGRAM NO. 6	

F 2683 2/67

Program No. 8

This program reduces the generating functions to simple coordinate transformation series.

The result of the Birkhoff normalization process is a sequence of generating functions which are time dependent polynomials dependent upon combinations of the old and new variables. In order to make direct use of these results in the area of trajectory prediction, the generating function must be operated on so as to produce independent series relating the old variables to the new ones and vice versa. This operation, known as an inversion, consists of a set of substitutions designed to evaluate one homogeneity level of the desired series at a time. The substitution is continued until the maximum order of the series is achieved. As an example, consider the following generating function;

$$W = 1X\eta^2 + 1X^2\eta$$

where X,Y are the old variables and ξ,η are the new variables. Inversion of function W up to third order results in the following four series:

$$X = 1\xi^1\eta^0 - 2\xi^1\eta^1 - 1\xi^2\eta^0 + 4\xi^1\eta^2 + 6\xi^2\eta^1 + 2\xi^3\eta^0$$

$$Y = 1\xi^0\eta^1 + 2\xi^1\eta^1 + 1\xi^0\eta^2 - 4\xi^1\eta^2 - 2\xi^2\eta^1$$

$$\xi = 1X^1Y^0 + 2X^1Y^1 + 1X^2Y^0 - 2X^1Y^2 - 4X^2Y^1$$

$$\eta = 1X^0Y^1 - 2X^1Y^1 + 6X^1Y^2 + 4X^2Y^1 + 2X^0Y^3$$

Supporting Subroutines

1. IDENT
2. IDENT1
3. INVER
4. SERIO
5. SEROP
6. SERPRT
7. STRAN

Input

1.	NIP(I), I = 1,72	Print control where I represents the subroutine number.
2.	NHMAX	Maximum homogeneity retained in the series representing coordinate transformations.
3.	ISET(4)	Maximum time level carried.
4.	NW	
	W	Generating function from Program No. 7.
	IW	

Printed Output

1.	NHMAX	Maximum homogeneity retained in the series representing the coordinate transformations.
2.	NW	
	W	Generating function from Program No. 7.
	IW	
3.	NS	
	S	Old coordinates X in terms of new coordinates Z.
	IS	
4.	NS	
	S	New coordinates Z in terms of old coordinates X.
	IS	

Punched Output

1.	X	Old coordinates in terms of new coordinates.
2.	Z	New coordinates in terms of old coordinates

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FORTRAN CODING FORM

PROGRAM No. 8		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER (1–5)	CONT (6)	FORTRAN STATEMENT (7–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
		NHMAX*ISET(4)	
		NW,W,IW FROM PROGRAM NO. 7	

F 2683 2 67

Program No. 9

This program combines the individual coordinate transformations into one total transformation.

Supporting Subroutines

1. IDENT
2. IDENT1
3. SERIO
4. SEROP
5. SERPRT
6. STRAN

Input

1. NIP(I), I = 1, 72	Print control where I represents the subroutine number.
2. N	Number of components in the state vector.
3. NSUB	Number of individual transformations to be combined.
4. NHMAX	Maximum homogeneity to be retained.
5. X	Individual coordinate transformation series relating the old variables to the new. From Program No. 6 and Program No. 8.
6. Z	Individual coordinate transformation series relating the new variables to the old. From Program No. 6 and Program No. 8.

Printed Output

1. N	Number of components in the state vector.
2. NSUB	Number of individual transformations to be combined.

Program No. 9 (continued)

3. NHMAX	Maximum homogeneity retained.
4. NX X IX	Initial X coordinates.
5. NZ Z IZ	Initial Z coordinates.
6. NY Y IY	Combined X coordinate transformation.
7. NY Y IY	Combined Z coordinate transformation.

Punched Output

1. X	Totaled old coordinates in terms of the new coordinates.
2. Z	Totaled new coordinates in terms of the old coordinates.

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PROGRAM	PROGRAM NO. 9		PUNCHING INSTRUCTIONS		PAGE	OF
			GRAPHIC		CARD ELECTRO NUMBER	
PROGRAMMER		DATE	PUNCH			

STATEMENT NUMBER (1–5)	CONT. (6)	FORTRAN STATEMENT (7–72)	SEQUENCE (Identification) (73–80)
		(NIP(I),I=1,72)	
	N	NSUB * NHMAX * ISET(4)	
		NX,X,IX FROM PROGRAM NO. 6	
		NZ,Z,IZ FROM PROGRAM NO. 6	
		NX,X,IX FROM PROGRAM NO. 8	
		NZ,Z,IZ FROM PROGRAM NO. 8	
		. . . . .	
		. . . . .	
		. . . . .	

F 2683 2/67

Program No. 10

This program numerically evaluates the solution of the normalized Hamiltonian equations, solved symbolically.

The Z coordinates (ξ_i, η_i) are evaluated after first calculating the initial values (ξ_{i0}, η_{i0}).

$$\xi_i = \xi_{i0} \cos (XLAM)_i \, t + \eta_{i0} \sin (XLAM)_i \, t$$

$$\eta_i = -\xi_{i0} \sin (XLAM)_i \, t + \eta_{i0} \cos (XLAM)_i \, t \quad .$$

The X coordinates are then evaluated at a specific value of t.

Supporting Subroutines

1. HAMNUM
2. IDENT
3. IDENT1
4. SERIO
5. SERNUM
6. SERPRT

Input

1.	NIP(I), I = 1,72	Print control where I represents the subroutine number.
2.	NV	Number of components in the X vector.
3.	NT	Number of time points into which the total time is divided.
4.	TT	Total problem time.
5.	XOC(I)	Initial condition for the i^{th} component.
6.	XLAM	Three eigenvalues of the linear system
7.	NX	
	X	X coordinates.
	IX	
8.	NZ	
	Z	Z coordinates.
	IZ	

Printed Output

1.	NV	Number of components in the X vector.
2.	NT	Number of time points into which the total time is divided.
3.	TT	Total time.
4.	XOC(I)	Initial condition for the i^{th} component.
5.	XLAM	Three eigenvalues of the linear system.
6.	NX	
	X	X coordinates.
	IX	
7.	NZ	
	Z	Z coordinates.
	IZ	
8.	DR	X coordinates numerically evaluated at each time point.

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FORTRAN CODING FORM

PROGRAM No. 10		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER	CONT	FORTRAN STATEMENT	SEQUENCE (Identification)
		(NIP(I),I=1,72)	
		NV = NT	
		TT	
		XOC(1) = XOC(2) = XOC(3)	
		XOC(4) = XOC(5) = XOC(6)	
		XLAM(1) = XLAM(2) = XLAM(3)	
		NX,X,IX FROM PROGRAM NO. 9	
		NZ,Z,IZ FROM PROGRAM NO. 9	

F 2683 2 '67

Program No. 11

This program numerically integrates the differential equations of a time-dependent Hamiltonian system. A Runge-Kutta numerical integration procedure is used.

Supporting Subroutines

1. EQUAT
2. IDENT
3. IDENT1
4. RKI
5. SERIO
6. SERNUM
7. SEROP
8. SERPRT

Input

1. NIP(I), I = 1, 72	Print control where I represents the subroutine number.
2. NV	Number of components in X vector.
3. NT	Number of time points into which the total time is divided.
4. NC	Number of integration cycles between print points.
5. TT	Total time.
6. XOC(I)	Initial condition of the i th component.
7. NH H IH	Hamiltonian series

Printed Output

1. NV NT NC TT XOC(I) NH H IH	Input data.

Program No. 11 (continued)

Printed Output

2. XDOT	Print-out of derivatives of H.
3. X	Print-out of X coordinates evaluated at time points.

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PROGRAM No. 11		PUNCHING INSTRUCTIONS		PAGE OF
		GRAPHIC		CARD ELECTRO NUMBER
PROGRAMMER	DATE	PUNCH		

STATEMENT NUMBER (1–5) | CONT. (6) | FORTRAN STATEMENT (7–72) | SEQUENCE (Identification) (73–80)

```
      (NIP(1),I=1,72)
      NV    NT    NC
      TT
      XOC(1)          XOC(2)
      .               .
      .               .
      NH,H,IH AS INPUT TO PROGRAM NO. 1
```

F 268 2 67

III. 3 Subroutines

	Routine	Description
1.	AMAT	Transforms a Hamiltonian system into a set of time dependent linear differential equations. The linearity of the differential equations is assured by using only the 2nd order terms in the Hamiltonian.
2.	CANMAT	Evaluates the matrix which is used in the coordinate transformation of a Hamiltonian to preserve the canonical property in the new variables.
3.	COMIN	Finds the inverse of a complex matrix using a Gaussian reduction technique.
4.	COMOP	Performs basic complex arithmetic such as multiplication, division.
5.	CONV	Transforms a series into a matrix element or vice-versa.
6.	DCUBIC	Obtains the roots of a cubic polynomial equation where the coefficient of X^3 is 1.
7.	DQRTC	Obtain the roots of a quartic polynomial equation where the coefficient of X^4 is 1.
8.	EIGEN	Finds the eigenvalues and eigenvectors of a real matrix.
9.	EIGVAL	Finds the eigenvalues of a complex matrix.
10.	EIGVEC	Finds the eigenvectors of a complex matrix when the eigenvalues are available.
11.	EMAT	Evaluates the exponential array E(t) which is used to multiply the characteristic matrix.
12.	EQUAT	Determines derivatives for the Runga-Kutta routine.
13.	GEN	Determines the generating junction of order IS for a Hamiltonian of order IS-1. The IS homogeneity part of the new Hamiltonian is also computed.
14.	HAMNUM	Performs the numerical evaluation of the X and Z coordinates at a specified time value.

	Routine	Description
15.	IDENT	Identifies series terms from the logical definition variable or forms the logical definition from specified bit variables. The terms must be of the form: 6 spatial and 1 exponential term.
16.	IDENT1	Same description as IDENT but must be used for variables of the form 7 exponential, 0 spatial and 1 time term.
17.	INVER	Accepts generating functions in terms of x and η and performs an inversion of variables to obtain ξ and η in terms of x and y or vice-versa.
18.	LINMAT	Solves a linear system of equations of the form AX =B.
19.	LINTRA	Performs a linear transformation of the spatial coordinates in a series.
20.	MATLOG	Finds the logarithm of a real matrix.
21.	MATMUL	Multiplies two complex matrices.
22.	MATIO	Punches or reads a matrix where each element is a series.
23.	MATOP	Performs matrix operations on square matrices where each element is a power series. Performs addition, subtraction, multiplication, and transfer from one array to another. Each element must have terms in the form specified by IDENT.
24.	MATOP1	Performs matrix operations such as addition, subtraction, multiplication, transfer, and selection of a particular time dependent order. Each element must have terms in the form specified in IDENT1.
25.	MATPRT	Prints out a matrix where each element is a series.
26.	NORM1	Determines a Hamiltonian normalized to order IS when a Hamiltonian normalized to order IS-1 is available. The generating function relating the old variables to the new variables is also included in the output.

Routine	Description
27. PROOT	Finds the roots of a polynomial with real coefficients.
28. RKI	Performs a Runga-Kutta integration.
29. SDIF1	This subroutine differentiates a series of the form - 7 exponential, 0 spatial, and one time term.
30. SERDET	Finds the determinant in series form of a matrix in which each element is a series.
31. SEREX1	Performs a Fourier expansion of a series containing non-commensurate eigenvalues. The resultant series is periodic with a period of 2π .
32. SERI0	Punches or reads in a series.
33. SERNUM	Computes the numerical value of a series at a specified time point.
34. SEROP	Performs basic power series operations such as addition, subtraction, multiplication, transfer, selection of particular homogeneity, truncation up to a particular homogeneity, and differentiation. Each term is of the form - 6 spatial and one exponential term.
35. SEROP1	Performs basic power series operations such as addition, subtraction, multiplication, and transfer. Each term is of the form 7 exponentials, 0 spatial, and one time term.
36. SERPRT	Prints out a series in a format that resembles the actual mathematical form.
37. SINT1	Performs the integration of a series of the form - 7 exponential, 0 spatial and one time term, from time zero to time T.
38. STRAN	Performs series transformations for Program 9.
39. TAY1	Generates one expansion series of the Taylor expansion where the expansion term has the following form:

$$\sum_{n=1}^{m} \sum_{I(1)=1}^{ndif} \cdots \sum_{I(n)=1}^{ndif} \frac{\partial^n A}{\partial[y_{I(1)}] \cdots \partial[y_{I(n)}]} \frac{\partial B}{\partial[x_{I(1)}]} \cdots \frac{\partial B}{\partial[x_{I(n)}]}$$

Routine	Description
40. TAY2	Generates the entire series of a particular type of Taylor expansion. The resulting series is composed of terms which all have the same homogeneity. A Taylor expansion of the following form is considered: $A\left(x, y + \frac{\partial B}{\partial x}\right)$ where A and B are power series. The order of $\frac{\partial B}{\partial x}$ must be higher than the order of y.
41. YMAT	Solves the linear time-dependent differential equation of the form $\dot{Y} = BY$ where B can be expanded in a Fourier series of the form: $B = B(0) + B(1) + B(2) + \ldots\ldots$

IV APPLICATION TO ELLIPTICAL RESTRICTED THREE-BODY PROBLEM

IV.1 Model Description and Related Equations

The frame of reference for the planar elliptical restricted three-body problem is a rectangular coordinate frame that rotates about its origin at the same speed as that with which the Earth and the Moon move in elliptical motion about their barycenter. The origin of this coordinate frame will represent the Earth-Moon barycenter, and these two masses, which will be designated m_1 and m_2 respectively, will be fixed in the coordinate frame if a unit of length is defined to be the distance which separates the Earth and the Moon. The equations of motion of a body m_o of insignificant mass in this coordinate system are [2]:

$$\ddot{x} - 2\dot{y} = \frac{1}{1 + e \cos t} \frac{\partial U}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{1}{1 + e \cos t} \frac{\partial U}{\partial y} \quad ,$$

where the potential function is

$$U = \frac{1 - m}{\rho_{o1}} + \frac{m}{\rho_{o2}} + \frac{1}{2}(x^2 + y^2) \quad ,$$

$$m = \frac{m_2}{m_1 + m_2} \quad ,$$

ρ_{o1} and ρ_{o2} are the distances between the body m_o and the massive bodies m_1 and m_2 respectively, measured in units of the distance between the two massive bodies, e is the eccentricity of the elliptical motion of m_2 relative to m_1, and dots denote differentiation with respect to the independent variable t, which here represents the true anomaly of the elliptical motion of m_2 about m_1. If we introduce the variables $u = \dot{x} - y$, $v = \dot{y} + x$, the equivalent Hamiltonian system of equations is

$$\dot{x} = \frac{\partial H}{\partial u}$$

$$\dot{y} = \frac{\partial H}{\partial v}$$

$$\dot{u} = -\frac{\partial H}{\partial x}$$

$$\dot{v} = -\frac{\partial H}{\partial y}$$

with the Hamiltonian

$$H = \tfrac{1}{2}(x^2 + y^2) + \tfrac{1}{2}(u^2 + v^2) + (uy - xv) - \frac{1}{1 + e\cos t}\, U(x, y) .$$

The origin is translated to the equilibrium point L_4 by the linear transformations:

$$x = \tfrac{1}{2} - m + q_1 , \quad y = \frac{\sqrt{3}}{2} + q_2 , \quad u = -\frac{\sqrt{3}}{2} + p_1 , \quad v = \tfrac{1}{2} - m + p_2 .$$

The Hamiltonian in the neighborhood of the equilibrium point $L_4 (q_1 = q_2 = p_1 = p_2 = 0)$ becomes:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}(q_1^2 + q_2^2) + (q_2 p_1 - q_1 p_2) - \frac{1}{1 + e\cos t}\, B(q)$$

where $$B(q) = \frac{1-m}{\rho_{o1}} + \frac{m}{\rho_{o2}} + \tfrac{1}{2}[q_1^2 + q_1(1-2m) + q_2^2 + \sqrt{3}\, q_2 + m^2 - m + 1]$$

where ρ_{o1} and ρ_{o2} are the distances between the body m_o and the massive bodies m_1 and m_2 respectively.

In implementing the numerical procedures, this Hamiltonian was expanded in a Taylor series through the fifth order of homogeneity. This series in turn was expanded in exponential form $(e^{ik\lambda t})$ to the second degree $(k = 0, \pm 1, \pm 2)$ in terms of the true anomaly t. The Hamiltonian contained, after the expansion, a total of 94 terms.

In starting the 2nd order normalization the expanded Hamiltonian is truncated to second order. The truncated Hamiltonian for the reduced three body problem is:

$$\begin{aligned} H = {} & 0.1244348731 q_1^2 - 0.626695381 q_2^2 + 1.0 p_1 q_2 - 1.0 q_1 p_2 \\ & + 0.5 p_1^2 + 0.5 p_2^2 - 0.126939203 q_1 q_2 \\ & + [0.01029375 q_1^2 + 0.03088125 q_2^2 + 0.34792411 q_1 q_2][e^{it} + e^{-it}] \\ & + [-0.0002825634 q_1^2 - 0.0008476903 q_2^2 - 0.0009550517 q_1 q_2][e^{2it} + e^{-2it}] . \end{aligned}$$

IV.2 Parameters and Trajectories

The accuracy of the solution obtained by normalization is affected by three parameters — the order of normalization N, the number of steps in the recursive solution for the characteristic matrix $\Phi(t)$, and the highest degree k retained in the Fourier expansion of the transformation matrix $P(t)$. This latter parameter appears also in the expansion of the Hamiltonian in terms of the independent variable t (true anomaly), but there we always retain terms through the second degree in k, as mentioned in section IV.1. A suitable choice of parameters is selected on the basis of acceptable accuracy of solutions, consistent with computational capability of the digital computer that is used. To attempt to normalize the Hamiltonian to 5th or 6th order, with many steps in the recursive solution for Φ and many terms retained in the Fourier expansion for $P(t)$, would require too much core space on a computer, even for the UNIVAC 1108 which we use. Accordingly, we judiciously selected, after preliminary investigation, certain combinations of parameters. We then compared the resulting solutions with the trajectory obtained by numerically integrating the same Hamiltonian equations which yielded the "normalized" solutions.

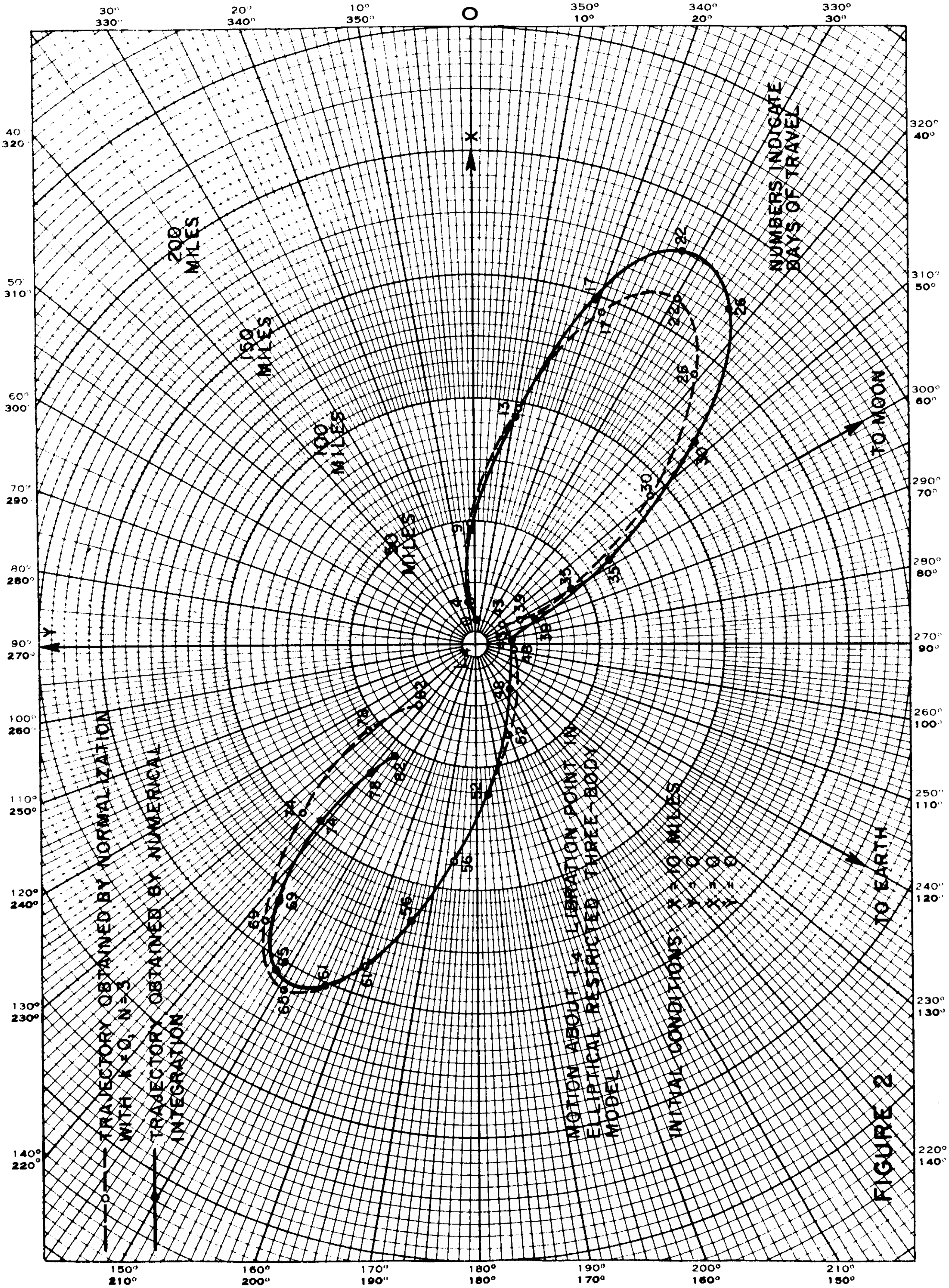
At the outset, it became apparent that two recursive steps are necessary and also sufficient in the solution for the characteristic matrix Φ. It is also necessary to retain some t-dependent terms in the Fourier expansion for P. That is, we must have $k > 0$. To ignore this time dependence is equivalent to neglecting the ellipticity of the motion of the moon about the earth. The effect of time-dependent (or k) terms in the Fourier expansion of P is illustrated in the trajectories of Figures 2 and 3. In Figure 2, the trajectory obtained by normalization, using as parameters $k = 0$, $N = 3$, departs noticeably from the numerically integrated trajectory by the 17th day. In fact this normalized curve is identical to the trajectory obtained by numerically integrating the circular restricted three-body equations. By contrast, in Figure 3, the curve obtained by normalization, using as parameters $k = 1$, $N = 3$, does not depart from the numerically integrated trajectory until the 30th day, approximately. Very little increase in accuracy is gained by retaining terms containing $k = 2$.

In the vicinity of the libration point L_4, the time dependence is more prominent than the effect of nonlinear terms. The marked difference between the normalized trajectories of Figures 2 and 3 is not at all evident when comparing the normalized trajectories of Figures 3 and 4. In both of the latter curves, $k = 1$; only the order of normalization is different, and the resulting trajectories are almost identical.

These effects of time dependence and order of normalization are apparently unchanged if the motion is expanded to a neighborhood about L_4 which is ten times the size of the neighborhood observed in Figures 2, 3, and 4. This is evident in the trajectories of Figures 5, 6, and 7. Our earlier statement about gaining little increase in accuracy by retaining terms containing $k = 2$, is also true in the larger neighborhood.

Before discussing the sources of errors in the normalized trajectories, we will conclude this section by pointing out two principal reasons for using the normalization procedure. First of all, of course, is the fact that the normalization process gives us approximate analytic solutions of the dynamical system. The approximate trajectories are expressed in terms of power series with known coefficients. Thus the motion of a particle or space vehicle can be studied analytically and the position and velocity computed for any instant, without having to compute the prior history of the particle. Qualitative behavior of the particle can be studied easily, to see the effects of changes in initial conditions. Secondly, because the independent variable appears explicitly in the normalized solution, the time required to compute normalized solutions is much less than the time required to compute corresponding trajectories by numerical integration of the equations of motion. In the normalized trajectories shown in the Figures 2-7, positions were calculated at 4-day intervals. Total computation time for the 80 days took about 3/4 of a minute, with variations of 10 seconds or so for different parametric (k, N) conditions. By contrast, numerical integration of each trajectory required from 4 to 11 minutes, depending upon the step size used in the integration. In these trajectories we found little difference in accuracy between step sizes of 3 hours (11 minutes total integration time) and 10 hours (4 minutes total itegration time). This type of comparison must be made prior to carrying out extensive numerical integration, and is time consuming in itself. And of course if we desire to know only the location of the particle at the end of 30 days, for example, we need

make only one calculation with the normalized equations. By numerical integration, we have to proceed step-by-step starting from the initial position.



TRAJECTORY OBTAINED BY NORMALIZATION WITH k=0, N=3
TRAJECTORY OBTAINED BY NUMERICAL INTEGRATION
MOTION ABOUT L4 LIBRATION POINT IN ELLIPTICAL RESTRICTED THREE-BODY MODEL
INITIAL CONDITIONS: X = 10 MILES
NUMBERS INDICATE DAYS OF TRAVEL
200 MILES
150 MILES
100 MILES
50 MILES
TO MOON
TO EARTH

FIGURE 2

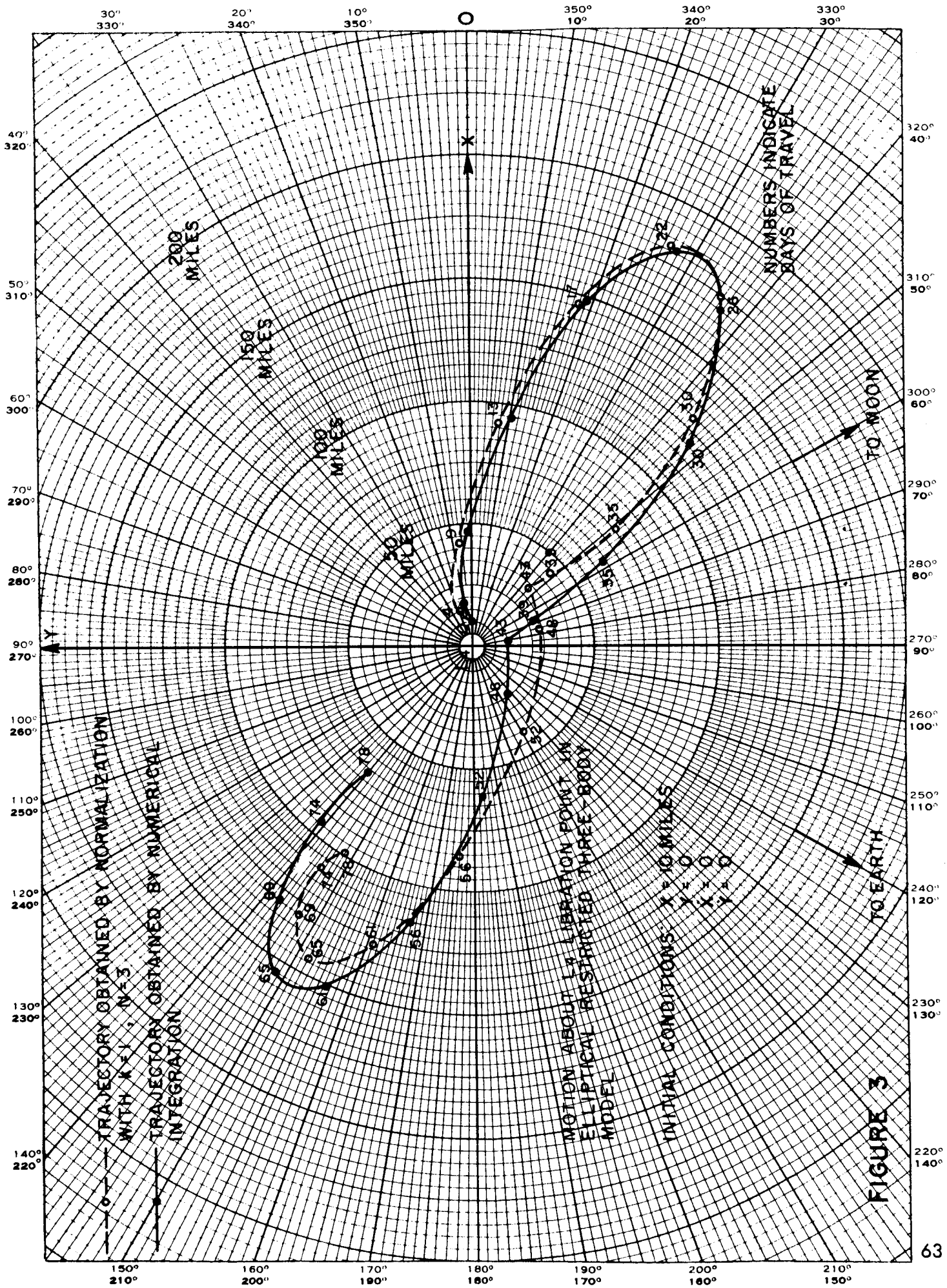

O
X
Y
TO MOON
TO EARTH
200 MILES
150 MILES
100 MILES
50 MILES
NUMBERS INDICATE DAYS OF TRAVEL
INITIAL CONDITIONS X=0 Y=0 X=0
MOTION ABOUT L4 LIBRATION POINT IN ELLIPTICAL RESTRICTED THREE-BODY MODEL
TRAJECTORY OBTAINED BY NUMERICAL INTEGRATION
TRAJECTORY OBTAINED BY NORMALIZATION WITH K=1, N=3

FIGURE 3

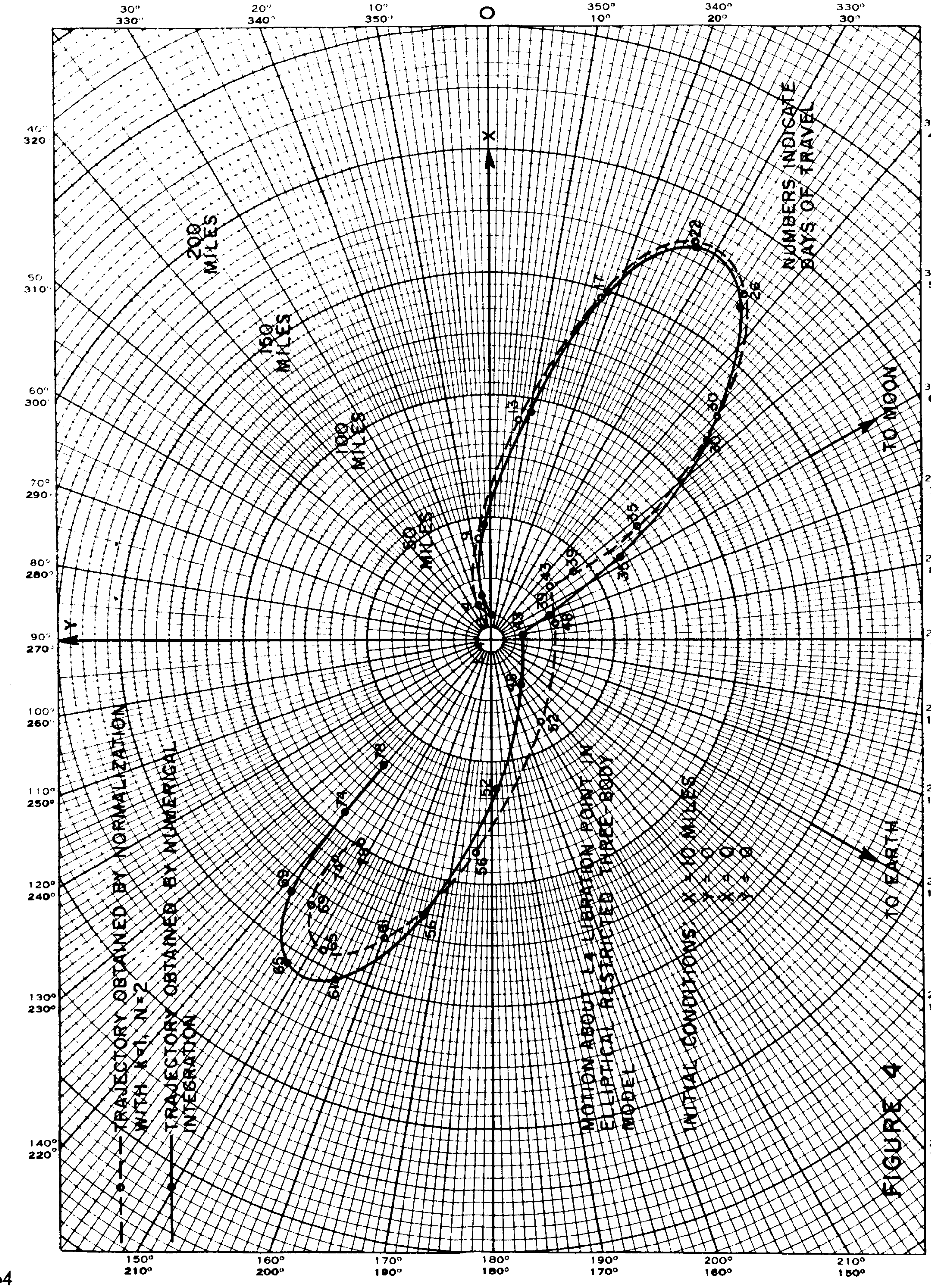

NUMBERS INDICATE DAYS OF TRAVEL
TO MOON
TO EARTH
200 MILES
150 MILES
100 MILES
50 MILES
X
Y
TRAJECTORY OBTAINED BY NORMALIZATION WITH K=1, N=2
TRAJECTORY OBTAINED BY NUMERICAL INTEGRATION
MOTION ABOUT L4 LIBRATION POINT IN ELLIPTICAL RESTRICTED THREE-BODY MODEL
INITIAL CONDITIONS X = 10 MILES Y = 0 X = 0 Y = 0

FIGURE 4

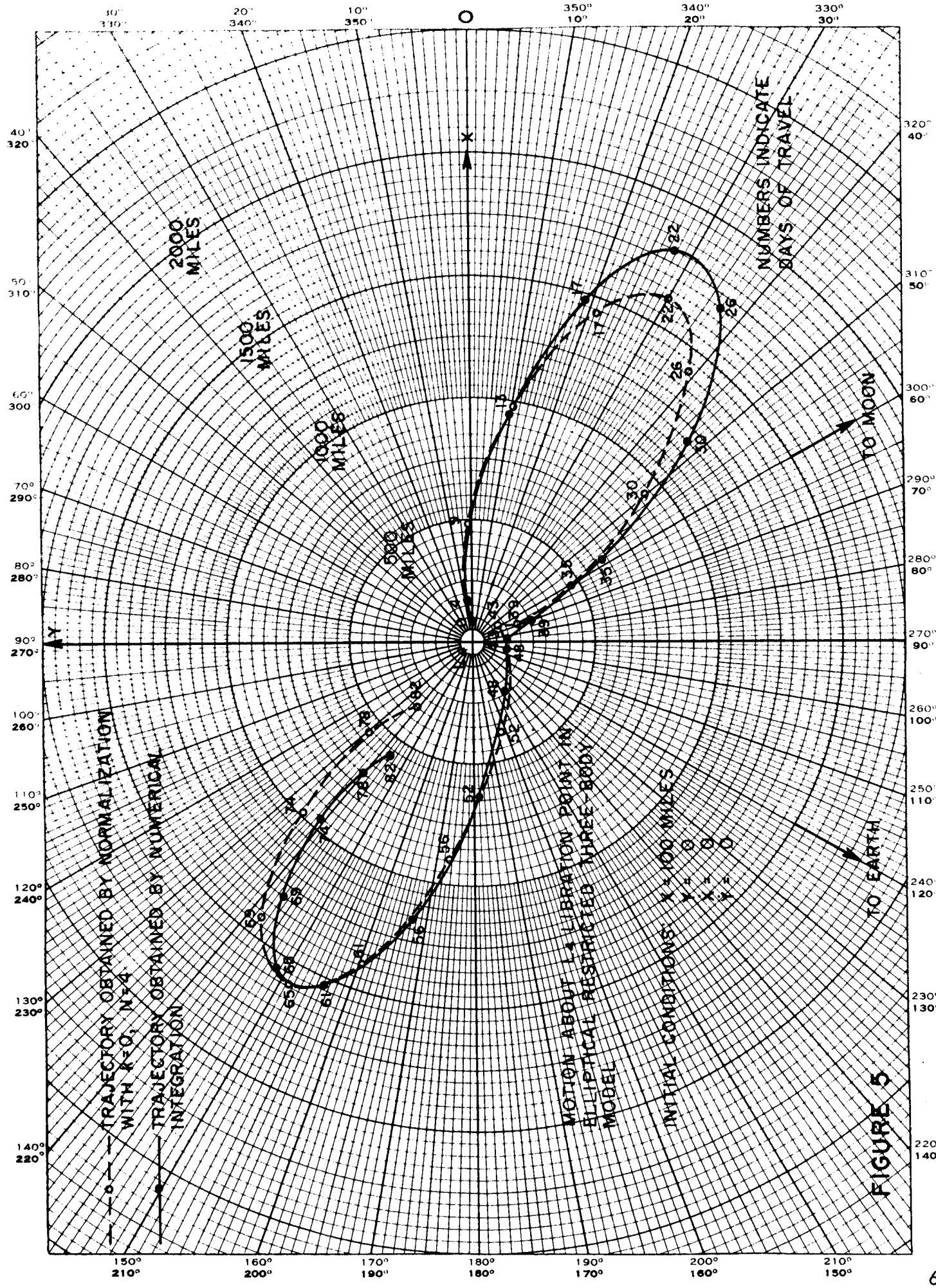

2000 MILES
1500 MILES
1000 MILES
500 MILES
NUMBERS INDICATE DAYS OF TRAVEL
TO MOON
TO EARTH
INITIAL CONDITIONS X = 100,000 MILES
MOTION ABOUT L1 LIBRATION POINT IN ELLIPTICAL RESTRICTED THREE-BODY MODEL
TRAJECTORY OBTAINED BY NUMERICAL INTEGRATION
TRAJECTORY OBTAINED BY NORMALIZATION WITH K=0, N=4
O
X
Y

FIGURE 5

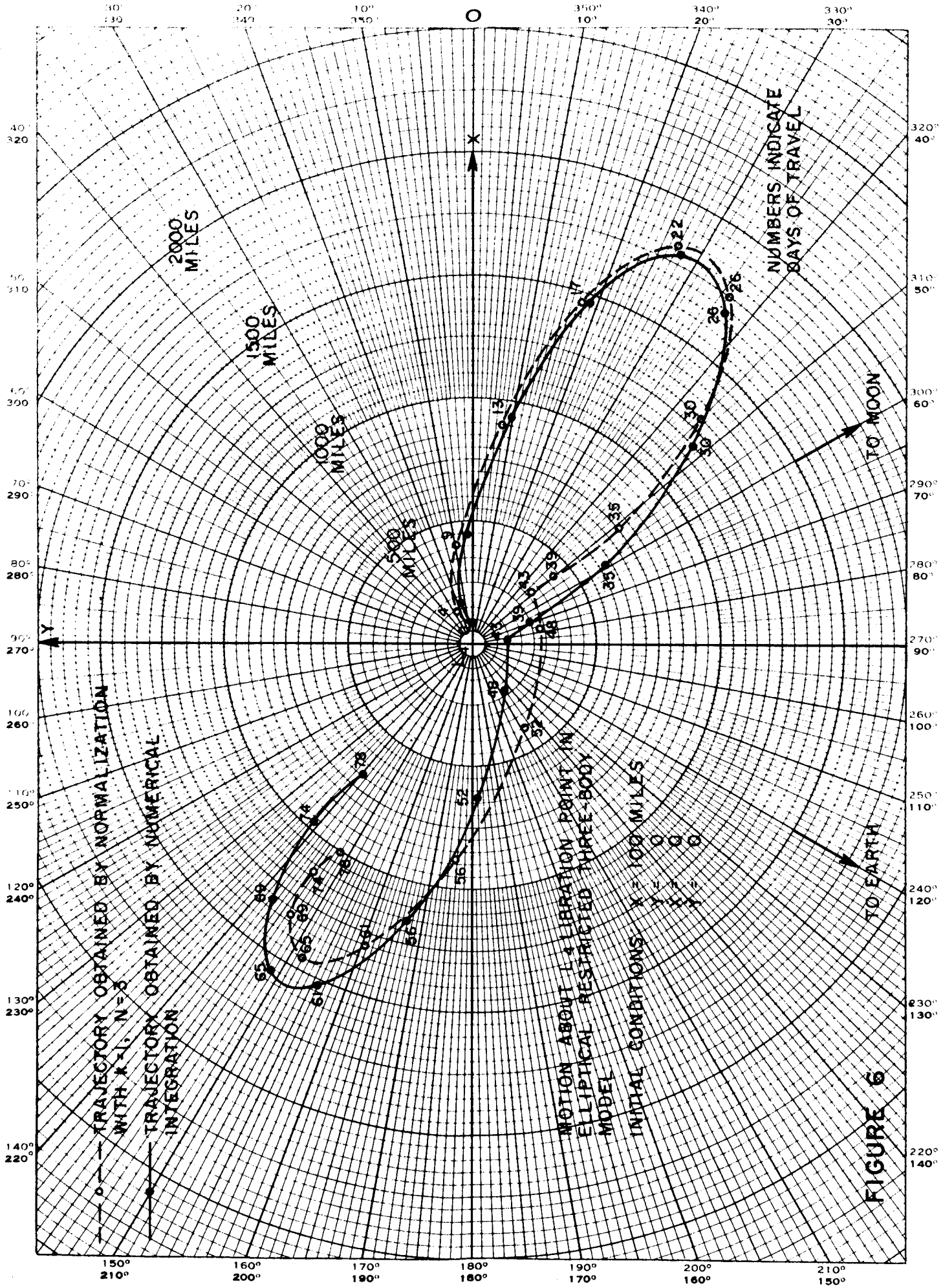

O
X
Y
2000 MILES
1500 MILES
1000 MILES
500 MILES
NUMBERS INDICATE DAYS OF TRAVEL
TO MOON
TO EARTH
MOTION ABOUT L4 LIBRATION POINT IN ELLIPTICAL RESTRICTED THREE-BODY MODEL
INITIAL CONDITIONS X = 100,000 Y = 0
TRAJECTORY OBTAINED BY NORMALIZATION WITH X-L4 N=3
TRAJECTORY OBTAINED BY NUMERICAL INTEGRATION

FIGURE 6

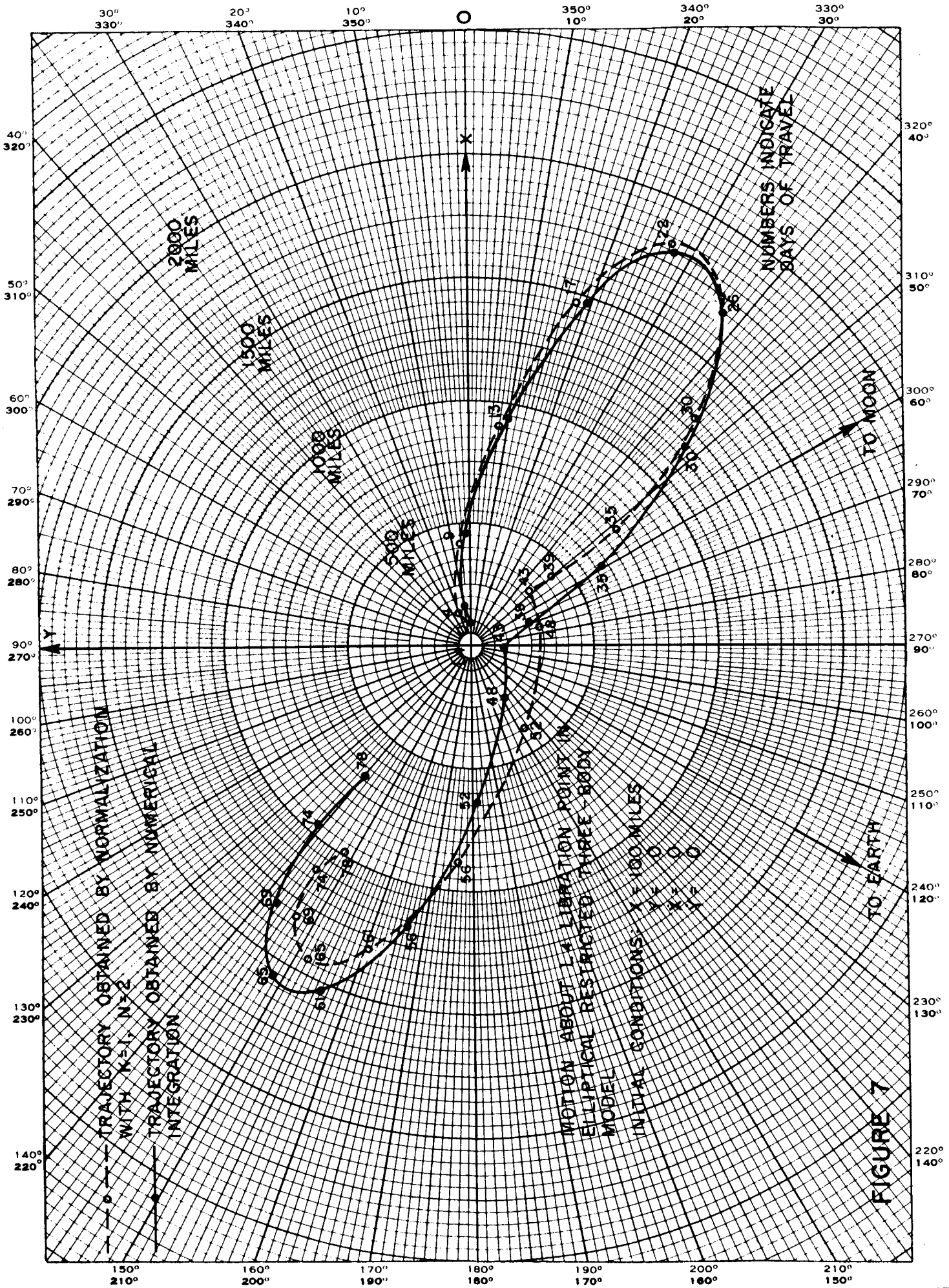

O
X
Y
TO MOON
TO EARTH
NUMBERS INDICATE DAYS OF TRAVEL
2000 MILES
1500 MILES
1000 MILES
500 MILES
MOTION ABOUT L4 LIBRATION POINT IN ELLIPTICAL RESTRICTED THREE-BODY MODEL
INITIAL CONDITIONS: Y = 100 MILES
TRAJECTORY OBTAINED BY NUMERICAL INTEGRATION
TRAJECTORY OBTAINED BY NORMALIZATION WITH K=1, N=2

FIGURE 7

IV.3 Error Sources

Even with time dependence and nonlinear effects included, the normalized solutions we have obtained ultimately diverge from the numerically integrated trajectories. This is because of truncation and round-off errors which occur at various stages of the normalization process.

1. The Birkhoff Normalization technique for solving the differential equation generated by a Hamiltonian system in the neighborhood of an equilibrium point requires that the Hamiltonian be expanded about that point, in terms consisting of state variable polynomials. It is at this point that new errors are introduced because of a number of factors:

   a) The Hamiltonian is expanded in terms of polynomials of state vector components. This expansion must be truncated.

   b) For time-dependent Hamiltonians (elliptical restricted three-body problem), the time-dependent contributions must be expanded in periodic functions and truncated.

2. Second order normalization for a time-dependent Hamiltonian involves a number of sequential processes. These processes are:

   a) Numerical computation of eigenvalues, producing round-off error.

   b) Use of a finite recursive technique to solve a system of linear time-dependent differential equations, which requires truncation. The solution to the equation is composed of periodic functions, whose coefficients are computed numerically. A great deal of numerical manipulation is required, hence the problem of round-off is significant.

3. As a result of each level of higher order normalization, a generating function and its corresponding transformed Hamiltonian is produced. The coefficients of terms in the generating functions and new Hamiltonians are subject to round-off error. Also, in the computation for the coefficients of terms in the new Hamiltonian, a small divisor is present which forces these numbers to become large. As a result of this division, the normalization can only be implemented for a limited number of orders.

4. The higher order normalization process yields generating functions, each of which implicitly relates two sets of coordinates. It is necessary to obtain explicit relations between these two coordinate sets. There are two techniques available for this purpose, namely:

a) numerical iteration

b) symbolic expansion,

The first of these does not produce algebraic functions relating coordinates, which is one of our objectives, so the second technique was studied here. This method does introduce truncation as well as round-off errors. The truncation errors can become excessive.

5. As a result of inverting each generating function the explicit representation of coordinates is available. The functions relating coordinates are combined to obtain one algebraic transformation to and from the original and final set. This computation is another source of round-off error.

6. The algebraic solution is a set of large time-dependent polynomials which have to be evaluated for each trajectory point. This evaluation again introduces round-off error.

To determine the relative magnitude of the errors due to these various sources, would require a systematic study which is beyond the scope of this report.

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